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# TWO EXAMPLES OF ZERO-DIMENSIONAL SETS IN PRODUCT SPACES

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Assuming the continuum hypothesis we give an example of a completely regular space $F$ without any dense finite-dimensional subspace whose square $F\times F$ contains a dense zero-dimensional subspace; the construction is based on an example of a metrizable separable space $H$ whose all uncountable subsets are infinite-dimensional, but whose square $H\times H$ contains an uncountable zero-dimensional subset.

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## 1. Terminology and notation

Our terminology follows [4, 7]. By the dimension we shall understand in this note the Menger–Urysohn inductive dimension ind [4, 7.1]. A space is countable-dimensional if it is a union of countably many zero-dimensional subspaces. By $\omega^*$ we denote the remainder $\beta\omega\setminus\omega$ of the Čech–Stone compactification of the natural numbers $\omega$, $I$ is the unit interval and $I^\infty$ is the Hilbert cube. The continuum hypothesis is abbreviated CH. The boundary of a set $S$ in a space $X$ is denoted by Bd $S$.

Given a subset $S$ in the product $X\times Y$ and a subset $A\subset X$ we denote by $S[A]$ the set $\{y\in Y: (x, y)\in S \text{ for some } x\in A\}$.

## 2. The examples

In this paper we give the following two examples.

**Example 2.1.** Assuming CH, there exists a metrizable separable space $H$ all whose uncountable subsets are infinite-dimensional, but whose square $H\times H$ contains an uncountable zero-dimensional subset.

**Example 2.2.** Assuming CH, there exists a completely regular space $F$ without any dense finite-dimensional subspace whose square $F\times F$ contains a dense zero-dimensional subspace.

Let us make a few remarks related to these examples.

**Remark 2.3.** Hurewicz [6], (see also [7, Section 28 IV, Remark (ii)]) proved that an existence of an uncountable metrizable separable space $X$ of which every uncountable subset is infinite-dimensional is equivalent to CH; Rančin [11] modified Hurewicz's argument to get, under CH, a metrizable separable space $X$ such that every uncountable subset of every finite power $X^n$ of $X$ is infinite-dimensional.

**Remark 2.4.** Assuming CH, Ciesielski [3] constructed a completely regular space without dense finite-dimensional subspaces and Malyhin [8] announced examples of similar type in some generic extensions; these examples have many additional properties (e.g. Ciesielski's space is hereditarily Lindelöf and Malyhin's examples are $\sigma$-compact). No such spaces were constructed so far in the realm of the usual set theory, cf. [1, Problem 12].

To obtain Example 2.1 we use zero-dimensional sets in $I^\infty\times I^\infty$ with "large projections". An existence of such sets is an easy consequence of a theorem of Bing [2] concerning hereditarily indecomposable separators in $I^\infty$ (see Lemma 3.1); another approach, which provides some stronger results in this direction, is discussed in Section 7.1.

Example 2.2 is based essentially on Example 2.1 and on some special properties of dense finite-dimensional subsets of $\omega^*\times I^\infty$, obtained under CH in Section 5. The space $F$ in Example 2.2 is a dense subspace of $\omega^*\times I^\infty$; under some alternative axiom for set theory, each dense subset of $\omega^*\times I^\infty$ has a dense zero-dimensional subspace, see Section 7.2.

## 3. Zero-dimensional sets in $I^\infty\times I^\infty$ with large projections

The following lemma will be used in constructions of the examples; a more general fact is discussed in Section 7.1.

**Lemma 3.1.** *There exists a zero-dimensional set $Z\subset I^\infty\times I^\infty$ such that for every countable-dimensional set $C\subset I^\infty$ the set $Z$ intersects the square $(I^\infty\setminus C)\times(I^\infty\setminus C)$.*

**Proof.** By a theorem of Bing [2, Theorem 3] there is a sequence $L_1, L_2,\ldots$ of hereditarily indecomposable continua [7, Section 48 V] in $I^\infty\times I^\infty$ such that every two disjoint closed sets in $I^\infty\times I^\infty$ can be separated by some $L_i$. Let us check that the zero-dimensional set $Z = I^\infty \times I^\infty \setminus (L_1 \cup L_2 \cup \cdots)$ has the required property. Let $C \subset I^\infty$ be any countable-dimensional set. There exists a nontrivial continuum $K$ in $I^\infty$ disjoint from $C$ (cf. [7, Section 28 IV, Theorem 4; 12, Lemma 5.2]) and it is enough to show that $K \times K \cap Z \neq \emptyset$. Assume on the contrary that $K \times K \subset L_1 \cup L_2 \cup \cdots$. By the Baire category theorem there exist some $L_k$ and nonempty open sets $U_1, U_2$ in the space $K$ such that $U_1 \times U_2 \subset L_k$. Each $U_i$ contains a nontrivial continuum $T_i$ [7, Section 47 II] and $T_1 \times T_2 \subset L_k$ which, however, contradicts the fact that $L_k$ is hereditarily indecomposable. □

**Remark 3.2.** We shall need a zero-dimensional set $Z \subset I^\infty \times I^\infty$ such that for each countable-dimensional $C \subset I^\infty$ the set $Z \cap ((I^\infty \setminus C) \times (I^\infty \setminus C))$ is dense in $I^\infty \times I^\infty$. To obtain such $Z$, let us pick up countably many pairwise disjoint topological copies $K_1, K_2, \ldots$ of $I^\infty$ in $I^\infty$, such that each nonempty open set in $I^\infty \times I^\infty$ contains some rectangle $K_i \times K_j$, for each pair $i, j$ let us choose a zero-dimensional set $Z_{ij} \subset K_i \times K_j$ given by Lemma 3.1 and let $Z = \bigcup_{i,j=1}^\infty Z_{ij}$.

## 4. Construction of Example 2.1

Let $Z$ be a zero-dimensional set described in Lemma 3.1. Assume CH and, following [6], let us arrange all finite-dimensional $G_\delta$-sets in $I^\infty$ into a sequence $G_1, G_2, \ldots, G_\xi, \ldots, \xi < \omega_1$, and let us choose for each $\xi$ points $x_\xi, y_\xi \in I^\infty \setminus \bigcup_{\eta<\xi} G_\eta$ such that $(x_\xi, y_\xi) \in Z$. The space $H = \{x_\xi : \xi < \omega_1\} \cup \{y_\xi : \xi < \omega_1\}$ has the required properties (cf. [7, Section 28 IV, Remark (ii)] and notice that $\{(x_\xi, y_\xi) : \xi < \omega_1\} \subset Z \cap (H \times H)$).

**Remark 4.1.** To obtain Example 2.2 we shall need the following modification of this reasoning. Let us consider the set $Z$ described in Remark 3.2 and, with the sets $G_\xi$ as above, for each $\xi < \omega_1$ let us choose countable sets $Q_{\xi 1}, Q_{\xi 2}, \ldots, Q_{\xi\xi}$ in $I^\infty$ such that

(1) $Q_{\xi\alpha}$ are pairwise disjoint for $\alpha \leqslant \xi < \omega_1$,
(2) $Q_{\xi\alpha} \cap \bigcup_{\eta<\xi} G_\eta = \emptyset$ for $\alpha \leqslant \xi < \omega_1$,
(3) $(Q_{\xi\xi} \times Q_{\xi\alpha}) \cap Z$ is dense in $I^\infty \times I^\infty$ for $\alpha \leqslant \xi < \omega_1$.

Let

$$H_\alpha = \bigcup_{\xi \geqslant \alpha} Q_{\xi\alpha}, \qquad H = \bigcup_{\alpha < \omega_1} H_\alpha.$$

The sets $H_\alpha$ are pairwise disjoint, by (1), and all uncountable subsets of $H$ are infinite-dimensional, by (2). Moreover, (3) guarantees that $(H_\alpha \times H_\beta) \cap Z$ is dense in $I^\infty \times I^\infty$ for all $\alpha, \beta < \omega_1$.

## 5. Inductive dimension of subsets of $\omega^* \times I^\infty$

In this section we prove the following proposition (to be used in the next section).

**Proposition 5.1.** *Assume CH. If $E \subset \omega^* \times I^\infty$ is a subset of dimension* $\operatorname{ind} E \leq n$ *in the product of the space of free ultrafilters on $\omega$ and the Hilbert cube, then there exists a nonempty open set $U \subset \omega^*$ such that* $\operatorname{ind} E[U] \leq 2n+1$, *where $E[U] = \{y \in I^\infty: (x, y) \in E$ for some $x \in U\}$.*

We begin with two simple observations about the space $\omega^* \times I^\infty$; the proofs are standard, but we include them for completeness sake.

**Lemma 5.2.** *Let $S \subset \omega^*$ be a nonempty open set and let $E \subset \omega^* \times I^\infty$ be a set whose projection onto the first axis is dense in S. There exists then a nonempty open set $T \subset S$ and a compact set $K \subset I^\infty$ such that $E \cap (T \times I^\infty)$ is a dense subset of $T \times K$.*

**Proof.** Assume that the assertion is not true. Then, using the fact that nonempty $G_\delta$-sets in $\omega^*$ have nonempty interior [4, Example 3.6.A], one can define by transfinite induction a decreasing sequence of nonempty open-and-closed subsets of $S$, $T_1 \supset T_2 \supset \cdots \supset T_\alpha \supset \cdots$ for $\alpha < \omega_1$, such that each compact set $K_\alpha = \overline{E[T_\alpha]}$ (see Section 1) is strictly contained in the intersection $\bigcap_{\beta<\alpha} K_\beta$. This, however, is impossible, as in $I^\infty$ no strictly decreasing sequence of type $\omega_1$ of closed sets exists. □

**Lemma 5.3.** *Let $W_1, W_2, \ldots$ be a sequence of open neighbourhoods of a point $(a, b)$ in $\omega^* \times I^\infty$. There exists then a nonempty open set $U \subset \omega^*$ and a sequence $V_1, V_2, \ldots$ of open neighbourhoods of the point b in $I^\infty$ such that $U \times V_i \subset \bar{W}_i$ and $U \times \operatorname{Bd} V_i \subset \operatorname{Bd} W_i$, for $i = 1, 2, \ldots$.*

**Proof.** The closed sets $\operatorname{Bd} W_i$ have empty interior, and hence the set of points $x$ in $\omega^*$ for which the vertical section $(\operatorname{Bd} W_i)[x]$ has nonempty interior in $I^\infty$ for some $i$ is of first category in $\omega^*$, cf. [7, Section 22 V]. Because nonempty $G_\delta$-sets in $\omega^*$ have nonempty interior, one can choose $p \in \omega^*$ such that $(p, b) \in W_i$ and $(\operatorname{Bd} W_i)[p]$ has empty interior for $i = 1, 2, \ldots$. Put

$$C_i = W_i[p] \quad \text{and} \quad D_i = I^\infty \setminus (\bar{W}_i)[p].$$

Since $I^\infty \setminus (C_i \cup D_i) = (\operatorname{Bd} W_i)[p]$, the set

$$C_i \cup D_i \text{ is dense in } I^\infty. \tag{1}$$

For each $i$ let us choose countably many open rectangles $A_{ij} \times C_{ij}$ and $B_{ij} \times D_{ij}$ in $\omega^* \times I^\infty$, $j = 1, 2, \ldots$, such that

$$p \in A_{ij}, \quad A_{ij} \times C_{ij} \subset W_i, \quad \bigcup_j C_{ij} = C_i, \tag{2}$$

$$p \in B_{ij}, \quad B_{ij} \times D_{ij} \subset I^\infty \setminus \bar{W}_i, \quad \bigcup_j D_{ij} = D_i, \tag{3}$$

and let

$$U = \mathrm{Int} \bigcap_{i,j=1}^{\infty} (A_{ij} \cap B_{ij}), \qquad V_i = \mathrm{Int}\, \bar{C}_i. \tag{4}$$

The open set $U$ is nonempty, each $V_i$ is an open neighbourhood of $b \in W_i[p] = C_i$ and, by (2) and (3)

$$U \times C_i \subset W_i, \quad U \times D_i \subset I^\infty \setminus \bar{W}_i, \qquad i = 1, 2, \ldots. \tag{5}$$

By (4) and (5), $U \times V_i \subset U \times \bar{C}_i \subset \bar{W}_i$. It remains to check that if $(x, y) \in U \times \mathrm{Bd}\, V_i$, then $(x, y) \in \mathrm{Bd}\, W_i$, i.e., since $(x, y) \in \bar{W}_i$, we have to verify that $(x, y) \notin W_i$. Let $G$ be any neighbourhood of $y$ in $I^\infty$: since $y \in \mathrm{Bd}\, V_i$, $G \setminus \bar{C}_i \neq \emptyset$ by (4), so by (1), $G \cap D_i \neq \emptyset$ and hence by (5), $\{x\} \times G \setminus \bar{W}_i \neq \emptyset$. □

**Proof of Proposition 5.1.** We shall check (assuming CH) by induction on $k$ the assertion: given a nonempty open-and-closed set $S \subset \omega^*$ and a subset $M$ of $S \times I^\infty$ with $\mathrm{ind}\, M \leqslant k$, there exists a nonempty open set $U \subset S$ with $\mathrm{ind}\, M[U] \leqslant 2k+1$.

The assertion is obvious for $k = -1$, let us assume its validity for $k < n$ and let $E \subset S \times I^\infty$ be a set with $\mathrm{ind}\, E = n$, $S$ being a nonempty open-and-closed set in $\omega^*$. We can assume that the projection of $E$ onto the first axis is dense in $S$ (otherwise, taking any nonempty set in $S$ disjoint from the projection, we are done). By Lemma 5.2, for some nonempty open set $T \subset S$ and a compactum $K \subset I^\infty$

$$E \cap (T \times I^\infty) \text{ is a dense subset of } T \times K. \tag{6}$$

We shall prove that

$$\mathrm{ind}\, E[U] \leqslant 2n+1 \quad \text{for some nonempty open } U \subset T. \tag{7}$$

Assume on the contrary that (7) is false. Using CH, let us arrange all $G_\delta$-sets of dimension $\leqslant 2n+1$ in the compactum $K$ into a transfinite sequence $G_1, G_2, \ldots, G_\alpha, \ldots, \alpha < \omega_1$. We shall choose for each $\alpha < \omega_1$ a nonempty open-and-closed set $U_\alpha \subset T$, countably many open sets $V_{\alpha 1}, V_{\alpha 2}, \ldots$ in $K$, a point $a_\alpha \in \omega^*$ and a point $b_\alpha \in K$ such that

$$a_\alpha \in \bigcap_{\beta<\alpha} U_\beta, \quad b_\alpha \notin G_\alpha, \quad (a_\alpha, b_\alpha) \in E, \tag{8}$$

$$\{V_{\alpha i}\}_{i=1}^{\infty} \text{ is a base of neighbourhoods at } b_\alpha \text{ in } K, \tag{9}$$

$$U_\alpha \subset \bigcap_{\beta<\alpha} U_\beta, \tag{10}$$

$$\mathrm{ind}\, B_{\alpha i} \leqslant 2n-1,$$

$$\text{where } B_{\alpha i} = \{y \in \mathrm{Bd}\, V_{\alpha i} : (x, y) \in E \text{ for some } x \in U_\alpha\}. \tag{11}$$

To make sure that such choice is possible, let us consider the following reasoning. Assume that we have given a nonempty open-and-closed set $A \subset T$ and a set $G \subset K$ of dimension $\leqslant 2n+1$. Since we assumed (7) is false, there exist $a \in A$ and $b \notin G$

such that $(a, b) \in E$. Let $H_1, H_2, \ldots$ be a base of neighbourhoods at the point $b$ in the space $K$. Since ind $E \leqslant n$, by (6) there are open neighbourhoods $W_1, W_2, \ldots$ of $(a, b)$ in $\omega^* \times K$ such that $W_i \subset A \times H_i$ and $\operatorname{ind}(\operatorname{Bd} W_i \cap E) \leqslant n-1$ for $i = 1, 2, \ldots$.

Next, we can use Lemma 5.3 to get a nonempty open-and-closed set $C \subset A$ and neighbourhoods $V_1, V_2, \ldots$ of $b$ in $K$ such that $C \times V_i \subset \bar{W}_i$ (hence $V_i \subset \bar{H}_i$) and $C \times \operatorname{Bd} V_i \subset \operatorname{Bd} W_i$, so in particular, $\operatorname{ind}(E \cap (C \times \operatorname{Bd} V_i)) \leqslant n-1$. Finally, by the inductive assumption, one can choose subsequently nonempty open-and-closed sets $U_1 \supset U_2 \supset \cdots$ in $C$ such that for $D_i = \{y \in \operatorname{Bd} V_i\colon (x, y) \in E$ for some $x \in U_i\}$, ind $D_i \leqslant 2(n-1)+1 = 2n-1$. Let $U$ be a nonempty open-and-closed subset of the intersection $\bigcap_{i=1}^{\infty} U_i$. The collection $\{V_i\}_{i=1}^{\infty}$ is an open base of neighbourhoods at $b$ in the space $K$ and for every $i$ the set $B_i = \{y \in \operatorname{Bd} V_i\colon (x, y) \in E$ for some $x \in U\} \subset D_i$ has dimension ind $B_i \leqslant 2n-1$.

We shall verify now that the conditions (8)–(11) yield a contradiction. Let us consider the subspace $L = \{b_\alpha\colon \alpha < \omega_1\}$ of the compactum $K$. The family $\{V_{\alpha i}\}_{i=1}^{\infty}$ is a base of neighbourhoods at $b_\alpha$ in $K$ (see (9)) and, by (8) and (11), $\operatorname{Bd} V_{\alpha i} \cap L \subset B_{\alpha i} \cup \{b_\xi\colon \xi \leqslant \alpha\}$. Since by (10), ind $B_{\alpha i} \leqslant 2n-1$ and $\alpha < \omega_1$, $\operatorname{ind}(\operatorname{Bd} V_{\alpha i} \cap L) \leqslant 2n-1+1 = 2n$ [7, Section 27 I]. It follows that $L$ is at most $(2n+1)$-dimensional and hence $L$ can be enlarged to a $G_\delta$-set in $K$ of dimension $\leqslant 2n+1$ [7, Section 27 IV], i.e., $L \subset G_\alpha$ for some $\alpha < \omega_1$. However, by (8), $b_\alpha \notin G_\alpha$—a contradiction which completes the proof. □

## 6. Construction of Example 2.2

To begin let us extract from Proposition 5.1 the following lemma.

**Lemma 6.1.** *Assume CH. Let $D$ be a dense subset of $\omega^*$, let $H$ be a subset of $I^\infty$ all whose uncountable subsets are infinite-dimensional (cf. Remark 2.3) and let $F\colon D \to H$ be any one-to-one correspondence. Then the graph $F \subset \omega^* \times I^\infty$ does not contain any dense finite-dimensional subspace.*

**Proof.** Let $E$ be a dense subspace of $F$. For each nonempty open set $U \subset \omega^*$, $E[U] = F(U \cap D)$ is an uncountable, hence infinite-dimensional subset of $H$, and the assertion follows from Proposition 5.1. □

We pass to the construction of Example 2.2. Let $Z$, $H_\alpha$ and $H$ be the sets described in Remark 4.1. Let us choose pairwise disjoint uncountable subsets $D_\xi$ of $\omega^*$ for $\xi < \omega_1$, such that each nonempty open subset of $\omega^*$ contains some $D_\xi$ and let $D = \bigcup_{\xi < \omega_1} D_\xi$. Let $F\colon D \to H$ be any one-to-one correspondence such that $F[D_\xi] = H_\xi$. Lemma 6.1 guarantees that all dense subsets of the graph $F \subset \omega^* \times I^\infty$ are infinite-dimensional. Let us verify that the square $F \times F$ contains a dense zero-dimensional subspace. We identify the product $(\omega^* \times I^\infty) \times (\omega^* \times I^\infty)$ with $(\omega^* \times \omega^*) \times (I^\infty \times I^\alpha)$ and we shall check that the zero-dimensional set $N = F \times F \cap ((\omega^* \times \omega^*) \times Z)$ is dense in $F \times F$. Given nonempty open sets $U \subset \omega^* \times \omega^*$ and

$W \subset I^\infty \times I^\infty$, consider $D_\alpha \times D_\beta \subset U$ and let $(y_1, y_2) \in (H_\alpha \times H_\beta) \cap Z \cap W$ (see Remark 4.1). There are points $x_1 \in D_\alpha$ and $x_2 \in D_\beta$ with $y_i = F(x_i)$, $i = 1, 2$, and then $(x_1, x_2, y_1, y_2) \in N \cap (U \times W)$.

## 7. Comments

**7.1.** The following statement is a generalization of Lemma 3.1 and, unless $E$ and $F$ are strongly infinite-dimensional compacta [9, VI.2], it requires a different approach.

**Proposition.** *Let $E$ and $F$ be uncountable-dimensional separable metrizable spaces and let $F$ be a continuous image of the irrationals. There exists then a zero-dimensional set $Z \subset E \times F$ such that for all countable-dimensional sets $A \subset E$ and $B \subset F$ the set $Z$ intersects the rectangle $(E \setminus A) \times (F \setminus B)$.*

**Proof.** We can assume that $E, F \subset I^\infty$. Let $p_i : I^\infty \to I$ be the projection onto the $i$th coordinate. There are an $n$ and a nontrivial interval $T \subset I$ such that for all $t \in T$ the set $p_n^{-1}(t) \cap E$ is uncountable-dimensional (indeed, if this was not the case, we could find for each $i$ a dense countable set $Q_i \subset I$ such that $p_i^{-1}(Q_i) \cap E$ is countable-dimensional, but then the set $E \subset \prod_{i=1}^\infty (I \setminus Q_i) \cup \bigcup_{i=1}^\infty (p_i^{-1}(Q_i) \cap E)$ would be countable-dimensional). Let the projection $p : I^\infty \times I^\infty \to I$ be defined by $p(x, y) = p_n(x)$. We shall use the following result from [10, Section 3]: there exists a zero-dimensional set $N \subset I^\infty \times I^\infty$ intersecting every zero-dimensional set $S \subset I^\infty \times I^\infty$ with $p(S) \supset T$. Let us check that the set $Z = N \cap (E \times F)$ has the required properties. Let $A \subset E$ and $B \subset F$ be countable-dimensional sets; one can find countable-dimensional $G_{\delta\sigma}$-sets $A^*$ and $B^*$ in $I^\infty$ with $A \subset A^*$ and $B \subset B^*$, cf. [7, Section 26 IV, Theorem 1]. The uncountable analytic set $F \setminus B^*$ contains a topological copy of the irrationals $P$ and let $f$ be a continuous map of $P$ onto $I^\infty \setminus A^*$, cf. [7, Section 37 I]. The graph $G = \{(f(y), y) : y \in P\}$ is zero-dimensional. Let $S = G \cap ((E \setminus A) \times (F \setminus B))$; we shall check that $p(S) \supset T$. Indeed, if $t \in T$, then, the set $p_n^{-1}(t) \cap E$ being uncountable-dimensional, there exists $y \in P$ with $f(y) \in (p_n^{-1}(t) \cap E) \setminus A^*$ and for $s = (f(y), y)$ we have $s \in S$ and $p(s) = t$. Now, $N$ intersects $S$ and hence $Z \cap ((E \setminus A) \times (F \setminus B)) \neq \emptyset$. $\square$

**Remark.** The result of Rančin quoted in Remark 2.3 shows that some additional assumption about $F$ in the proposition is necessary.

**7.2.** Let (R) be the following statement: the intersection of any family of cardinality $\aleph_1$ of open subsets of $\omega^*$ is either empty or has nonempty interior; the statement (R) is independent of ZFC axioms, see [5, Axiom 26 C, 11.D].

The following proposition shows that some extra axioms for set theory are necessary in Lemma 6.1.

**Proposition.** *Assume the hypothesis* (R). *Then every dense subspace $F$ of $\omega^* \times I^\infty$ contains a dense zero-dimensional subspace.*

**Proof.** Let $I^\infty = \bigcup_{\xi<\omega_1} N_\xi$, where $N_\xi$ are zero-dimensional sets. Let $F$ be a dense subspace of $\omega^* \times I^\infty$; it is enough to check that for every nonempty open rectangle $A \times B$ in $\omega^* \times I^\infty$ there is a zero-dimensional set $Z \subset F \cap (A \times B)$ whose closure has nonempty interior in $\omega^* \times I^\infty$. Let $B_1, B_2, \ldots$ be an open base in $B$. We shall choose successively nonempty open-and-closed subsets of $A$, $D_1 \supset D_2 \supset \cdots$, compact nonempty sets $K_i \subset B_i$ and ordinal numbers $\alpha_i < \omega_1$ such that $K_i \cap K_j = \emptyset$ for $i \neq j$, and for $M_i = N_{\alpha_i} \cap K_i$ the set $F \cap (D_i \times M_i)$ is dense in $D_i \times K_i$. If for some $i$ we get a compactum $K_i$ with nonempty interior, we stop, as in this case $Z = F \cap (D_i \times M_i)$ has the required property. Let us assume that $D_j$ and $K_j$ for $j \leqslant n$ have been defined and that the compacta $K_j$ have empty interior. We shall choose $D_{n+1}$, $K_{n+1}$ and $\alpha_{n+1}$ (the first step is the same as the $(n+1)$th one). Let $W$ be a nonempty open set whose closure is contained in $B_{n+1} \setminus (K_1 \cup \cdots \cup K_n)$ and let $X = \{x \in D_n:$ there exists $y \in N_\alpha \cap W$ with $(x, y) \in F\}$. The union $\bigcup_{\alpha<\omega_1} X_\alpha$ covers the projection of $F \cap (D_n \times W)$ onto $\omega^*$, hence the hypothesis (R) guarantees that for some $\alpha < \omega_1$ there is a nonempty closed-and-open set $C$ contained in $\bar{X}_\alpha$; we let $\alpha_{n+1} = \alpha$. By Lemma 5.2, there is a nonempty closed-and-open set $D_{n+1} \subset C$ and a compact set $K_{n+1} \subset \bar{W}$ such that, for $M = N_\alpha \cap W$, the set $F \cap (D_{n+1} \times M)$ is a dense subset of $D_{n+1} \times K_{n+1}$.

Now, let $D$ be a nonempty open-and-closed subset of the intersection $\bigcap_{i=1}^\infty D_i$. Since zero-dimensional sets $M_i$ have pairwise disjoint closures, their union $N = \bigcup_{i=1}^\infty M_i$ is zero-dimensional. Because each $F \cap (D \times M_i)$ is dense in $D \times K_i$, the zero-dimensional set $F \cap (D \times N)$ is dense in $D \times B$, which completes the proof. □

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